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# On Superposition and Entropy in Quantum Dynamics

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*Abstract*

Some considerations on the existence of nonlinear reversible dynamical evolutions for isolated quantum systems are developed within the Hilbert model with the aid of the superposition principle and the law of entropy.

## 1. *Introduction*

In this note the problem of the existence of nonlinear reversible dynamical evolutions for isolated quantum systems is investigated. To this end a definition of the superposition principle in a unified form is employed. More precisely, the dynamics is introduced by means of a continuous one-parameter group of permutations of the statistical operators of the system preserving the unified superposition (dynamical group). A corresponding Heisenberg picture is defined. Dynamical groups that give rise to the same Heisenberg picture are called equivalent. Using standard results a simple condition is shown to be sufficient in order to have a dynamical group equivalent to a unitary one.

The main result of the paper is the construction of a class of mathematical examples of nonlinear dynamical groups that are both equivalent to a unitary one and satisfy the condition mentioned. The physical meaning of these examples is investigated assuming that the processes a dynamical group describes to be truly reversible. On the basis of this assumption they are ruled out as unphysical, since they require that the entropy of the system is to be constant in time.

The considerations are developed in the Hilbert model, in the line of the results of Jauch (1968) and Piron (1964), taking into account reformulations and contributions due to Varadarajan (1968) and Pool (1968).

## 2. *Time Evolution: The Role of Superposition*

We associate to the physical system the lattice $\mathscr{L}(H)$ of all the closed subspaces of a separable complex Hilbert space $H$ ($\dim H \geqslant 3$) and the set $M$ of all the $\sigma$ additive probability measures (states) on $\mathscr{L}(H)$. According to the Piron theorem (Piron, 1964), every element of $\mathscr{L}(H)$ represents a class of equivalent yes–no experiments on the physical system. The elements of $M$ represent the set of the preparing procedures pertaining to the physical system. By the Gleason theorem (Gleason, 1957) every state on $\mathscr{L}(H)$ has the form $\rho(a) = \mathrm{Tr}P^a\rho$ where $a \in \mathscr{L}(H)$, $P^a$ is the orthogonal projection with range $a$ and $\rho \in K(H)$, $K(H)$ being the set of all positive, trace class operators with trace 1 (statistical operators) on $H$.

In the following, $K(H)$ and $M$ will be identified. The number $\rho(a)$ gives the probability of the outcome "yes" for a test of the class $a$ when the system has been prepared with the procedures corresponding to the state $\rho$. The set of the pure states of $K(H)$ (one-dimensional projections) and the set of the atoms of $\mathscr{L}(H)$ (rays of $H$) are denoted by $P(H)$ and $A(H)$, respectively.

The two sets $S_1(a) = \{\rho \in K(H)\colon \rho(a) = 1\}$ and $L(D) = \{b \in \mathscr{L}(H)\colon \rho(b) = 1 \forall \rho \in D\}$ $[a \in \mathscr{L}(H), D \subset K(H)]$ are the basic elements for the following considerations (Pool, 1968; Berzi and Zecca, 1974).

*Definition 1.* A state $\rho$ is said to be a superposition of the states in $D \subset K(H)$ if $L(\rho) \supset L(D)$ (Berzi and Zecca, 1974).

The formulation of the superposition principle is equivalent (Berzi and Zecca, 1974) to the one originally proposed in Varadarajan (1968). It has the advantage of being applicable not only to the pure states, but also to the statistical operators of the system. Moreover, it gives a unified form of both the concept of statistical mixture and the concept of quantum superposition. It is an intrinsic formulation that adapts also to schemes more general than the Hilbert one (Varadarajan, 1968; Berzi and Zecca, 1974; Gorini and Zecca, 1975).

The map $D \to \bar{D} = \{\rho \in K(H)\colon L(\rho) \supset L(D)\}$ is a closure operation (Birkhoff, 1973) (closure under superposition) on the subsets of $K(H)$ such that the family of the corresponding closed subsets of $K(H)$ is $\{S_1(a)\colon a \in \mathscr{L}(H)\}$. Moreover $S_1(\vee_\alpha a_\alpha) = \overline{\cup_\alpha S_1(a_\alpha)}$ for every family $\{a_\alpha\} \subset \mathscr{L}(H)$, where $\vee_\alpha a_\alpha$ is the least closed subspace of $H$ containing the $a_\alpha$'s. These assertions follow applying to the Hilbert model the results of Gorini and Zecca (1975).

*Definition 2.* A dynamical group is a continuous one-parameter group of permutations of $K(H)$: $t \to A_t, A_{t+s} = A_t A_s, A_0 = \mathbb{I}$ such that

(i) $L(\rho) \supset L(D) \Rightarrow L(A_t\rho) \supset L(A_t D), \qquad \forall\, t \in \mathbb{R}$

(ii) $t \to (A_t\rho)(a)$ is a continuous map $\forall \rho \in K(H)$, $\forall a \in \mathscr{L}(H)$

A dynamical group is assumed to describe the reversible dynamical processes (Schrödinger picture) in the idealized situation of a strictly isolated quantum physical system. Condition (i) of Definition 2 has a simple operative meaning.

Suppose a yes–no experiment performed on the physical system that has been prepared in the state $\rho$ gives the answer "yes" with certainty whenever it has been found certainly true on every state of $D$. Then this relation (superposition) is invariant under time translation. It is a weaker assumption than what is generally admitted,[1] because only the certainty value of the probability is required to be preserved. Condition (ii) is plausible on physical grounds.

It is easily seen that the standard unitary time evolution is an example of dynamical group. Suppose indeed $\mathrm{Tr} P^a(U_t \sigma U_t^+) = 1 \;\forall\; \sigma \in D$ where $t \to U_t$ is a weakly continuous one-parameter group of unitary operators on $H$. By setting $P^c = U_t^+ P^a U_t$ there follows $c \in L(D) \subset L(\rho)$. Hence $a \in L(U_t \rho U_t^+)$.

In this connection the question arises whether a unitary time evolution represents the most general dynamical group or not. As will be seen, a dynamical group need not be linear even under special assumptions.

## 3. *A Class of Dynamical Groups*

Definition 2 has at least two equivalent formulations.

*Lemma.* Let $t \to A_t$ be a dynamical group. Then condition (i) of Definition 2 is equivalent to every one of the following conditions:

(i) $A_t \bar{D} = \overline{A_t D}$, $\quad \forall t \in \mathbb{R}$, $\quad D \subset K(H)$

(ii) $A_t S_1(a) = \overline{A_t S_1(a)}$, $\quad \forall t \in \mathbb{R}$, $\quad \forall a \in \mathscr{L}(H)$

*Proof.* Lemma 2.2 of Gorini and Zecca (1975) still works in the Hilbert model and gives the equivalence of (i) with (i) of Definition 2. Trivially (i) $\Rightarrow$ (ii) because $S_1(a)$ is closed $\forall a \in \mathscr{L}(H)$. To prove (ii) $\Rightarrow$ (i) notice first that $\forall D \subset K(H) \exists$ $b \in \mathscr{L}(H)$ such that $\bar{D} = S_1(b)$. Hence, by (ii) $\overline{A_t \bar{D}} = A_t \bar{D} \supset \overline{A_t D}$. On the other hand also $A_{-t} = (A_t)^{-1}$ maps closed sets to closed sets. Hence from $A_{-t}(\overline{A_t D}) \supset D$, by taking closure, one gets $\overline{A_t D} \supset A_t \bar{D}$.

Using the spectral decomposition of a statistical operator one can easily verify the identity $S_1(a) = \{P^a\}\; \forall a \in A(H)$. Condition (ii) of the Lemma then enables one to show that $A_t$ is again a permutation when restricted to the set $P(H)$ of the pure states.

Given a dynamical group $t \to A_t$, one can uniquely define a one-parameter group $t \to Z_t^A$ of order preserving permutations of $\mathscr{L}(H)$ by setting

$$A_{-t} S_1(a) = S_1(Z_t^A(a))$$

This is possible by means of the result (ii) of the Lemma and the fact that $S_1(a) = S_1(b)$ i.f.f. $a = b\, [a, b \in \mathscr{L}(H)]$. (Compare with Gorini and Zecca, 1975.) It is easily seen that $Z_t^A$ preserves lattice join and meet for any family of elements of $\mathscr{L}(H)$ and also that it gives a permutation when restricted to the set $A(H)$ of atoms of $\mathscr{L}(H)$.

The one-parameter group $t \to Z_t^A$ is assumed to represent the *Heisenberg picture* induced by the *Schrödinger picture* $t \to A_t$.

[1] See, for instance, Jauch (1968). Compare also with Mackey (1963).

The dynamical groups $t \to A_t$ and $t \to B_t$ are said to be *equivalent* if $Z_t^A = Z_t^B\ \forall t \in \mathbb{R}$.

In the following proposition, which is straightforward consequence of a standard mathematical result, a sufficient condition is given to have a dynamical group equivalent to a unitary one.

*Proposition.* Let $t \to A_t$ be a dynamical group satisfying the condition

(iii) $\sum_i \alpha_i P_i = \sum_K \beta_K Q_K \Rightarrow \sum_i \alpha_i A_t P_i = \sum_K \beta_K A_t Q_K, \qquad \{P_i\}, \{Q_K\} \subset P(H),$

$\{\alpha_i\}, \{\beta_K\} \subset [0, 1]$

with

$$\sum_i \alpha_i = \sum_K \beta_K = 1$$

Then there exists a dynamical group $t \to B_t$, such that

(a) $B_t\rho = U_t \rho U_t^+\ \forall \rho \in K(H),\ \forall t \in \mathbb{R}$, where $t \to U_t$ is a strongly continuous one-parameter group of unitary operators on $H$;

(b) $Z_t^A = Z_t^B\ \forall t \in \mathbb{R}$.

*Proof.* From the very definitions one gets the identity

$$A_t P^a = P^{Z_t^A(a)}\ \forall a \in A(H)$$

Define now $B_t$: $K(H) \to K(H)$ by means of $B_t\rho = \Sigma_i \alpha_i A_t P_i$ for $\rho = \Sigma_i \alpha_i P_i \in K(H)$ where the $\alpha_i$'s are the (possibly repeated) eigenvalues of $\rho$ and $\{P_i\} \subset P(H)$. The definition of $B_t$ is consistent with possible degeneracy of $\rho$ because of assumption (iii). It follows that $B_t$ is a convex permutation of $K(H)$. Using results of Kadison (1965) and taking into account that $A_t$ and $B_t$ act on the pure states in the same way, one gets $B_t\rho = U_t\rho U_t^+\ \forall \rho \in K(H)$, where $t \to U_t$ is a weakly continuous (Bargmann, 1954; Varadarajan, 1968, Vol. II, Chap. XI) [hence strongly (Yosida, 1971)] one-parameter group of unitary operators on $H$. (Antiunitary operators are not considered because of the group property.) Hence $t \to B_t$ is a unitary dynamical group. The point (b) follows writing every $a \in \mathscr{L}(H)$ as a union of atoms: $a = \vee_i a_i$ and from

$$S_1(Z_{-t}^B(a)) = B_t \overline{U_i S_1(a_i)} = \overline{U_i B_t S_1(a_i)} = \overline{U_i A_t S_1(a_i)} = S_1(Z_{-t}^A(a))$$

where a property of the closure under superposition and condition (i) of the Lemma have been used.

The proposition implies that every dynamical group $t \to A_t$ that satisfies condition (iii) is such that the corresponding $t \to B_t$ verifies the equivalence of the Schrödinger and Heisenberg picture, that is, $(B_t\rho)(a) = \rho(Z_t^B(a))\forall a \in \mathscr{L}(H)$ $\forall t \in \mathbb{R}$. Hence the generator of the related $t \to U_t$ can be interpreted as the Hamiltonian of the system.

## 4. *Nonlinear Examples and the Role of Entropy*

The very definition of $t \to B_t$ in the proposition suggests how to construct nonlinear examples of dynamical groups starting from a unitary one. Suppose, indeed, $\dim H < \infty$ and consider any continuous family $t \to f_t$ of (nonadditive) positive, continuous, invertible functions defined on the positive real axis with the properties

(i) $f_t(xy) = f_t(x) f_t(y)$

(ii) $f_{t+s}(x) = f_t(f_s(x)), \qquad f_t^{-1}(x) = f_{-t}(x)$

$\forall x, y > 0$, $\forall t, s \in \mathbb{R}$ [e.g., $f_t(x) = x^{e^t}$]. If $\rho = \Sigma_i \gamma_i P_i$ is the spectral decomposition of a statistical matrix, define $\Lambda_t$: $K(H) \to K(H)$ by means of

(iii) $\Lambda_t \rho = \Sigma_i \dfrac{f_t(\gamma_i)}{\Sigma_K f_t(\gamma_K)} P_i$

Taking into account the uniqueness of the spectral decomposition and the properties of the one-parameter group of functions, one can verify (with some calculations) that $t \to \Lambda_t$ is a one-parameter group of bijections of $K(H)$ onto itself that satisfy condition (ii) of Definition 2 and condition (iii) of the Proposition.

We recall that for the least element $\Lambda L(D) = \Lambda_{b \in L(D)} b$ $[D \subset K(H)]$ of the dual principal (lattice) ideal $L(D)$ it holds that $\Lambda L(D) \equiv [D]$, where $[D]$ is the linear span of the ranges of all the statistical matrices $\sigma \in D$ (see Gorini and Zecca, 1975, sec. 4). With this notation, condition (i) of Definition 2 reads: $[\rho] \leqslant [D] \Rightarrow [A_t \rho] \leqslant [A_t D]$ $\forall t \in \mathbb{R}$ $[\rho \in K(H), D \subset K(H)]$. This condition is satisfied by the one-parameter group $t \to \Lambda_t$ because $[\Lambda_t \rho] = [\rho]$ $\forall \rho \in K(H)$, $\forall t \in \mathbb{R}$.

If $t \to U_t$ is a unitary dynamical group, then $t \to U_t \Lambda_t$ is a nonlinear dynamical group equivalent to $t \to U_t$.

Now the problem arises of the physical interpretation of such examples.

We suppose that the entropy $S(t)$ at time $t$ of a quantum system may be defined in terms of the statistical operator $\rho_t$ of the system at time $t$:

$$S(t) = -\mathrm{Tr}(\rho_t \log \rho_t)$$

The assumption that a dynamical group describes reversible physical processes implies the law of the constancy of entropy:

(iv)
$$\frac{dS(t)}{dt} = 0, \qquad \forall t \in \mathbb{R}$$

It is a trivial matter to check that this law is verified when the time evolution is given by a unitary dynamical group.

It will now be seen that the same law rules out as unphysical the given class of nonlinear examples. Consider, indeed, the state

$$\rho_t = [f_t(\alpha)P + f_t(1-\alpha)Q] / (f_t(1-\alpha))$$

which, according to the previous construction, is the time evolved of the statistical matrix $\rho$ of rank 2, whose spectral decomposition is $\rho = \alpha P + (1 - \alpha)Q$. Imposing condition (iv) on the entropy calculated in the given state (for mathematical details see Kato, 1966), after assuming the existence of the derivative $f_t'$ with respect to $t$ of $t \to f_t$, one gets

$$f_t'(\alpha)f_t(1-\alpha) - f_t(\alpha)f_t'(1-\alpha) = 0, \qquad \forall t \in \mathbb{R}$$

that is, $f_t(\alpha) = g(\alpha)f_t(1-\alpha)$, where $g(\alpha)$ is independent of $t$. The property (i) of the one-parameter group $t \to f_t$ then implies $f_t[\alpha/(1-\alpha)] = g(\alpha)$, $\forall t \in \mathbb{R}$, $0 < \alpha < 1$. Hence $f_t(\alpha) = f(\alpha)$, $\forall t \in \mathbb{R}$, where $f$ is a fixed function. Finally the group property with the invertibility of $f$ implies $f = f^2 = 1$.

Suppose now a one-parameter group of functions $t \to f_t$ exists such that the definition of $\Lambda_t$ in (iii) makes sense also when $H$ is infinite dimensional. Proceeding as above one gets that also in this case $f_t$ must be the identity function for every time $t$.

## 5. *Concluding Remarks*

In this note a possible characterization of the reversible dynamical processes that the physical system undergoes has been proposed in terms of general physical motivations based on Definition 2 and the law of entropy. In this connection some considerations arise: First of all the mathematical problem is open of determining necessary and sufficient conditions in order to have a dynamical group equivalent to a unitary one. Secondly, the question arises whether nonlinear dynamical groups exist that are equivalent to a unitary one and that verify the law of the entropy. It would likewise be interesting to know whether there exist nonlinear examples compatible with the law of entropy but not equivalent to a unitary dynamical group.

In any case, a possible indication for the construction of nonlinear examples could be that of proceeding as in (iii) of the previous section, but modifying non-linearly not only the eigenvalues but also the eigenprojections of the statistical operator.

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